PROBLEMS FOR OVEALL CONTEST

Problem 1. Show that for any integer $n \geq 3$ there are infinitely many irreducible polynomials of the form

$$x^n + (6a-1)x^2 + (7b-3)x + 25c \in \mathbb{Z}[x]$$

for some $a, b, c \in \mathbb{Z}[x]$.

Problem 2. The matrix $\mathbf{A}$ is defined by $a_{ij} = 1$, when $i + j$ is even and $a_{ij} = 0$, when $i + j$ is odd. The order of the matrix is $2n$. Show that

$$\|\mathbf{A}\|_F = \|\mathbf{A}\|_\infty = n,$$

where $\|\mathbf{A}\|_F$ is the Frobenius norm, and that

$$\sum_{k=1}^{\infty} \left(\frac{1}{2n}\right)^k \mathbf{A}^k = \frac{1}{n}\mathbf{A}.$$

Problem 3. Let X and Y be two Hilbert spaces, with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, and the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Consider a bounded operator T mapping from X to Y, with its adjoint operator given by T^*. For any $\beta > 0$ and $z \in Y$, consider the minimization

$$\min_{f\in X} J(f) := \frac{1}{2}\|Tf - z\|_Y^2 + \frac{\beta}{2}\|f\|_X^2\,,$$

and write its minimizer f as $f(\beta)$, and its minimal value function as $F(\beta)$, i.e., $F(\beta) = J(f(\beta))$.

(1) Prove $f(\beta) \in X$ satisfies

$$(Tf,\ Tg)_Y + \beta\,(f,\ g)_X = (z,\ Tg)_Y \quad \text{for all} \quad g \in X.$$

(2) Prove the n-th derivative $w = f^{(n)}(\beta) \in X$ satisfies

$$(Tw, Tg)_Y + \beta\,(w,\ g)_X = -n\,(f^{(n-1)}(\beta),\ g)_X \quad \text{for all} \quad g \in X.$$

(3) Prove the first and second derivatives of $F(\beta)$ are given by

$$F'(\beta) = \frac{1}{2}\|f(\beta)\|_X^2, \quad F''(\beta) = (f(\beta), f'(\beta))_X\,.$$

(4) If $z \notin \ker T^*$, prove $F(\beta)$ is strictly monotonically increasing and strictly concave.